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► To cite this version:

Jean-Yves Enjalbert, Hoang Ngoc Minh. Combinatorial study of colored Hurwitz polyzêtas. 2011.
hal-00704926

HAL Id: hal-00704926

<https://hal.science/hal-00704926>

Preprint submitted on 6 Jun 2012

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Combinatorial study of colored Hurwitz polyzêtas

June 6, 2012

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abstract

A combinatorial study discloses two surjective morphisms between generalized shuffle algebras and algebras generated by the colored Hurwitz polyzêtas. The combinatorial aspects of the products and co-products involved in these algebras will be examined.

1 Introduction

Classically, the Riemann zêta function is $\zeta(s) = \sum_{n>0} n^{-s}$, the Hurwitz zêta function is $\zeta(s; t) = \sum_{n>0} (n - t)^{-s}$ and the colored zêta function is $\zeta_q(s) = \sum_{n>0} q^n n^{-s}$, where q is a root of unit. The three previous functions are defined over $\mathbb{Z}_{>0}$ but can be generalized over any *composition* (sequence of positive integers) $\mathbf{s} = (s_1, \dots, s_r)$, like, respectively, the Riemann polyzêta function $\zeta(\mathbf{s}) = \sum_{n_1>\dots>n_r>0} n_1^{-s_1} \dots n_r^{-s_r}$, the Hurwitz polyzêta function $\zeta(\mathbf{s}; \mathbf{t}) = \sum_{n_1>\dots>n_r>0} (n_1 - t_1)^{-s_1} \dots (n_r - t_r)^{-s_r}$ and the colored polyzêta function $\zeta_q(\mathbf{s}) = \sum_{n_1>\dots>n_r>0} q^{i_1 n_1} \dots q^{i_r n_r} n_1^{-s_1} \dots n_r^{-s_r}$, with q a root of unit and $\mathbf{i} = (i_1, \dots, i_r)$ a composition. These sums converge when $s_1 > 1$.

To study simultaneously these families of polyzêtas, the colored Hurwitz polyzêtas, for a composition $\mathbf{s} = (s_1, \dots, s_r)$ and a tuple of complex numbers $\xi = (\xi_1, \dots, \xi_r)$ and a tuple of parameters in $] - \infty; 1[$, $\mathbf{t} = (t_1, \dots, t_r)$, are defined by [6]

$$\text{Di}(\mathbf{F}_{\xi, \mathbf{t}}; \mathbf{s}) = \sum_{n_1>\dots>n_r>0} \frac{\xi_1^{n_1} \dots \xi_r^{n_r}}{(n_1 - t_1)^{s_1} \dots (n_r - t_r)^{s_r}}. \quad (1)$$

Note that, for $l = 1 \dots, r$, the numbers ξ_l are not necessary roots of unity q^{i_l} . We are working, in this note, with the condition

$$(E) \quad \forall i, \left| \prod_{k=1}^i \xi_k \right| \leq 1 \quad \text{and} \quad t_i \in]-\infty; 1[.$$

Hence, $\text{Di}(\mathbf{F}_{\xi, \mathbf{t}}; \mathbf{s})$ converges if $s_1 > 1$. We note \mathcal{E} the set of \mathbb{C} -tuples verifying (E).

These polyzêtas are obtained as special values of iterated integrals¹ over singular differential 1-forms introduced in [10]. As iterated integrals, they are encoded by words or by non commutative formal power series [10] and are used to construct bases for asymptotic expanding [14] or symbolic integrating fuchian differential equations [11] exactly or approximatively [8]. The meromorphic continuation of the colored Hurwitz polyzêtas² is already studied in [5, 6]. In our studies, we constructed an integral representation³ of colored Hurwitz polyzêtas and a *distribution* treating *simultaneously two singularities* and our methods permit to make the meromorphic continuation *commutatively* over the variables s_1, \dots, s_r [5, 6]. Moreover, [6] gives another way to obtain the meromorphic continuation thanks to *translation equations* [4]. Our methods give the structure of multi-poles [5] (Theorem 4.2) and two ways to calculate algorithmically the multi-residus⁴.

In this note, in continuation with our previous works [10, 11, 12, 13, 5, 6], we are focusing on *Hopf algebra*, for a class of products as *minusstuffle* (\sqcup), *multistuffle* (\sqcup), \dots , and in particular for the *new* product *duffle* (\boxplus), obtained as “tensorial product” of \sqcup and the well known *stuffle* (\boxplus), of symbolic representations of these polyzêtas (see Definition 2.1 and Proposition 2.1 bellow).

2 Combinatorial objects

2.1 Some products and their algebraic structures

Let X be an encoding alphabet and the free monoid over X is denoted by X^* . The *length* of any word $w \in X^*$ is denoted by $|w|$ and the unit of X^* is denoted by 1_{X^*} . For any unitary commutative algebra A , a formal power series S over X with coefficients in A can be written as the infinite sum $\sum_{w \in X^*} \langle S | w \rangle w$. The set of polynomials (resp. formal power series) over X with coefficients in A is denoted by $A\langle X \rangle$ (resp. $A\langle\langle X \rangle\rangle$). The set of degree 1 monomials is $AX = \{ax/a \in A, x \in X\}$.

Definition 2.1 We note \mathcal{P} the set of products \star over $A\langle X \rangle$ verifying the conditions :

¹They are presented as generalized Nielsen polylogarithms in [10] (Definition 2.3) and as generalized Lerch functions in [12] (Definition 3).

²See also references and a discussion about meromorphic continuation of Riemann polyzêtas in [5].

³This integral representation is obtained by applying successively the polylogarithmic transform [10]. It is an application of *non* commutative convolution as shown in [9] (Section 2.4). Other integral representations can be also deduced easily by change of variables, for example $t = zr$ and then $r = e^{-u}$ [5].

⁴Other meromorphic continuations can also be obtained by Mellin transform as already done in [17] or by classical estimation on the imaginary part [7] but these later work reccursively, depth by depth, and the commutativity of this process over the variables s_1, \dots, s_r must be proved. Unfortunately, the structure of multi-poles as well as multi-residus are missing in both works [7, 17]. In [16], to make the meromorphic continuation (giving the expression of non positive integers multi-residus via a generalization of Bernoulli numbers – but not of *all* multi-residus) of the specialization at roots of unity of colored Hurwitz polyzêtas $\text{Di}(\mathbf{F}_{\xi, \mathbf{t}}; \mathbf{s})$, the author bases on the integral representation, on the contours, of the multiple Hurwitz-Lerch which corresponds *mutatis mutandis* to the integral representation of generalized Lerch functions introduced earlier in [5] (Corollary 3.3).

(i) the map $\star : A\langle X \rangle \times A\langle X \rangle \rightarrow A\langle X \rangle$ is bilinear,

(ii) for any $w \in X^*$, $1_{X^*} \star w = w \star 1_{X^*} = w$,

(iii) for any $a, b \in X$ and $u, v \in X^*$,

$$au \star bv = a(u \star bv) + b(au \star v) + [a, b](u \star v),$$

where $[\cdot, \cdot] : AX \times AX \rightarrow AX$ is a function verifying :

$$(S1) \quad \forall a \in AX, [a, 0] = 0,$$

$$(S2) \quad \forall (a, b) \in (AX)^2, [a, b] = [b, a],$$

$$(S3) \quad \forall (a, b, c) \in (AX)^3, [[a, b], c] = [a, [b, c]].$$

Example 1 (see [18]) Product of iterated integrals.

The shuffle is a bilinear product such that :

$$\begin{aligned} \forall w \in X^* \quad w \sqcup 1_{X^*} = 1_{X^*} \sqcup w = w \quad \text{and} \\ \forall (a, b) \in X^2, \forall (u, v) \in X^{*2}, \quad au \sqcup vb = a(u \sqcup bv) + b(au \sqcup v). \end{aligned}$$

For example, for any letter x_0, x and x' in X ,

$$x_0 x' \sqcup x_0^2 x = x_0 x' x_0^2 x + 2x_0^2 x' x_0 x + 3x_0^3 x' x + 3x_0^3 x x' + x_0^2 x x_0 x'.$$

Example 2 (see [15]) Product of quasi-symmetric functions.

Let X be an alphabet indexed by \mathbb{N} .

The stuffle is a bilinear product such that :

$$\begin{aligned} \forall w \in X^*, \quad w \sqcup 1_{X^*} = 1_{X^*} \sqcup w = w \quad \text{and} \\ \forall (x_i, x_j) \in X^2, \forall (u, v) \in X^{*2}, \\ x_i u \sqcup x_j v = x_i (u \sqcup x_j v) + x_j (x_i u \sqcup v) + x_{i+j} (u \sqcup v). \end{aligned}$$

In particular, with the alphabet $Y = \{y_1, y_2, y_3, \dots\}$,

$$(y_3 y_1) \sqcup y_2 = y_3 y_1 y_2 + y_3 y_2 y_1 + y_3 y_3 + y_2 y_3 y_1 + y_5 y_1.$$

Example 3 ([3]) Product of large multiple harmonic sums.

Let X be an alphabet indexed by \mathbb{N} .

The minus-stuffle is a bilinear product such that :

$$\begin{aligned} \forall w \in X^*, \quad w \sqcup 1_{X^*} = 1_{X^*} \sqcup w = w \quad \text{and} \\ \forall (x_i, x_j) \in X^2, \forall (u, v) \in X^{*2}, \\ x_i u \sqcup x_j v = x_i (u \sqcup x_j v) + x_j (x_i u \sqcup v) - x_{i+j} (u \sqcup v). \end{aligned}$$

Example 4 ([6]) Product of colored sums.

Let X be an alphabet indexed by a monoid (\mathcal{I}, \times) .

The mulstuffle is a bilinear product such that :

$$\begin{aligned} \forall w \in X^* \quad w \sqcup 1_{X^*} = 1_{X^*} \sqcup w = w \quad \text{and} \\ \forall (x_i, x_j) \in X^2, \forall (u, v) \in X^{*2}, \\ x_i u \sqcup x_j v = x_i (u \sqcup x_j v) + x_j (x_i u \sqcup v) + x_{i \times j} (u \sqcup v). \end{aligned}$$

For example, with X indexed by \mathbb{Q}^* ,

$$x_{\frac{2}{3}} x_{-1} \sqcup x_{\frac{1}{2}} = x_{\frac{2}{3}} x_{-1} x_{\frac{1}{2}} + x_{\frac{2}{3}} x_{\frac{1}{2}} x_{-1} + x_{\frac{2}{3}} x_{-\frac{1}{2}} + x_{\frac{1}{2}} x_{\frac{2}{3}} x_{-1} + x_{\frac{1}{3}} x_{-1}.$$

Remark 2.1 Thanks to the one-to-one correspondence $(i_1, \dots, i_r) \mapsto x_{i_1} \dots x_{i_r}$ between tuples of \mathcal{I} and word over X , the calculus of $x_{\frac{2}{3}} x_{-1} \sqcup x_{\frac{1}{2}}$ can be written as $(\frac{2}{3}, -1) \sqcup (\frac{1}{2}) = (\frac{2}{3}, -1, \frac{1}{2}) + (\frac{2}{3}, \frac{1}{2}, -1) + (\frac{2}{3}, \frac{-1}{2}) + (\frac{1}{2}, \frac{2}{3}, -1) + (\frac{1}{2}, -1)$.

Example 5 ([6]) Product of colored Hurwitz polyzêtas.

Let Y and E be two alphabets and consider the alphabet $A = Y \times E$ with the concatenation defined recursively by $(y, e).(w_Y, w_E) = (yw_Y, ew_E)$ for any letters $y \in Y$, $e \in E$, and any word $w_Y \in Y^*$, $w_E \in E^*$. The unit of the monoid A^* is given by $1_{A^*} = (1_{Y^*}, 1_{E^*})$. If Y is indexed by \mathbb{N} and E by a monoid (\mathcal{I}, \times) , the duffle is a bilinear product such that $\forall w \in A^*$, $w \sqcup 1_{A^*} = 1_{A^*} \sqcup w = w$, $\forall (y_i, y_j) \in Y^2, \forall (e_l, e_k) \in E^2, \forall (u, v) \in A^{*2}$, $(y_i, e_l).u \sqcup (y_j, e_k).v = (y_i, e_l).(u \sqcup (y_j, e_k).v) + (y_j, e_k).((y_i, e_l).u \sqcup v) + (y_{i+j}, e_{l \times k}).(u \sqcup v)$.

Proposition 2.1 The shuffle, the stuffle, the minus-stuffle and the multistuffle are elements of \mathcal{P} , with respectively, $[x_i, x_j] = 0$, $[x_i, x_j] = x_{i+j}$, $[x_i, x_j] = -x_{i+j}$, $[x_i, x_j] = x_{i \times j}$ for any letters x_i and x_j of X .

The duffle is in \mathcal{P} , with $[(y_i, e_l), (y_j, e_k)] = (y_{i+j}, e_{l \times k})$ for all y_i, y_j in Y , e_l, e_k in E .

Proposition 2.2 Let $\star \in \mathcal{P}$, then $(A\langle X \rangle, \star)$ is a commutative algebra.

Proof. We just have to show the commutativity and the associativity of \star .

To obtain $w_1 \star w_2 = w_2 \star w_1$ for all w_1, w_2 in X^* , we use an induction on $|w_1| + |w_2|$. It is true when $|w_1| + |w_2| \leq 1$ thanks to (i) since w_1 or w_2 is 1_{X^*} . The equality (iii), the condition (S2) and the commutative of $+$ give the induction. In the same way, an induction on $|w_1| + |w_2| + |w_3|$ gives $w_1 \star (w_2 \star w_3) = (w_1 \star w_2) \star w_3$ thanks to (iii) and (S3). \square

If we associate to each letter of X an integer number called weight, the weight of a word is the sum of the weight of its letters. In this case X is graduated.

In [15], Hoffman works over $\overline{X} = X \cup \{0\}$ with $[\cdot, \cdot] : \overline{X} \times \overline{X} \rightarrow \overline{X}$ and call *quasi-product* any product in \mathcal{P} with the additional condition :

(S4) Either $[a, b] = 0$ for all a, b in X ; or the weight of $[a, b]$ is the sum of the weight of a and the weight of b for all a, b in X .

Example 6 1. The shuffle is a quasi-product.

2. Let X be an alphabet indexed by \mathbb{N} and define the weight of x_i , $i \in \mathbb{N}$, by i . Then the stuffle is a quasi-product.

Theorem 2.1 ([15]) If X is graduated and has a quasi-product \star , then $(A\langle X \rangle, \star)$ is a commutative graduated A -algebra..

We can define (i) a comultiplication $\Delta : A\langle X \rangle \rightarrow A\langle X \rangle \otimes A\langle X \rangle$,
(ii) a counit $\epsilon : A\langle X \rangle \rightarrow A$,

by : $\forall w \in X^*$, $\Delta w = \sum_{uv=w} u \otimes v$ and $\epsilon(w) = \begin{cases} 1 & \text{if } w = 1_{X^*} \\ 0 & \text{otherwise.} \end{cases}$

The coproduct Δ is coassociative so $(A\langle X \rangle, \Delta, \epsilon)$ is a coalgebra.

Lemma 2.1 For any $w \in X^*$ and $x \in X$, $(x \otimes 1_{X^*})\Delta w + 1_{X^*} \otimes xw = \Delta xw$.

Proof. $\forall w \in X^*, \forall x \in X, \Delta xw = \sum_{uv=xw} u \otimes v = \sum_{u'v=w} xu' \otimes v + 1_{X^*} \otimes xw$

so $\Delta xw = x \otimes 1_{X^*} \left(\sum_{u'v=w} u' \otimes v \right) + 1_{X^*} \otimes xw = (x \otimes 1_{X^*})\Delta w + 1_{X^*} \otimes xw. \quad \square$

Proposition 2.3 If $\star \in \mathcal{P}$, then $(A\langle X \rangle, \star, \Delta, \epsilon)$ is a bialgebra.

Remember that \star acts over $A\langle X \rangle \otimes A\langle X \rangle$ by $(u \otimes v) \star (u' \otimes v') = (u \star u') \otimes (v \star v')$.

Proof. ϵ is obviously a \star -homomorphism. It still has to be show $\Delta(w_1) \star \Delta(w_2) = \Delta(w_1 \star w_2)$ over X^* . This equality is true if w_1 or w_2 is equal to 1_{X^*} .

Assume now that $\Delta(u) \star \Delta(v) = \Delta(u \star v)$ for any word u and v such that $|u| + |v| \leq n$, $n \in \mathbb{N}$, and let w_1 and w_2 be in X^* with $|w_1| + |w_2| = n + 1$. We note $w_1 = au$ and $w_2 = bv$, with a and b two letters of X , u and v two words of X^* . Thus, by definition, $\Delta w_1 = \sum_{u_1 u_2 = u} au_1 \otimes u_2 + 1_{X^*} \otimes au$ and $\Delta w_2 = \sum_{v_1 v_2 = v} bv_1 \otimes v_2 + 1_{X^*} \otimes bv$.

$$\begin{aligned}
& \Delta(w_1) \star \Delta(w_2) \\
&= \sum_{u_1 u_2 = u, v_1 v_2 = v} (au_1 \star bv_1) \otimes (u_2 \star v_2) + \sum_{u_1 u_2 = u} (au_1) \otimes (u_2 \star bv) \\
&+ \sum_{v_1 v_2 = v} (bv_1) \otimes (au \star v_2) + 1_{X^*} \otimes (au \star bv) \\
&= \sum_{u_1 u_2 = u, v_1 v_2 = v} (a(u_1 \star bv_1) \otimes (u_2 \star v_2) + b(au_1 \star v_1) \otimes (u_2 \star v_2) \\
&+ ([a, b](u_1 \star v_1)) \otimes (u_2 \star v_2)) + \sum_{u_1 u_2 = u} (au_1) \otimes (u_2 \star bv) \\
&+ \sum_{v_1 v_2 = v} (bv_1) \otimes (au \star v_2) + 1_{X^*} \otimes a(u \star bv) \\
&+ 1_{X^*} \otimes b(au \star v) + 1_{X^*} \otimes [a, b](u \star v) \\
&= \sum_{u_1 u_2 = u, v_1 v_2 = v} a(u_1 \star bv_1) \otimes (u_2 \star v_2) + \sum_{u_1 u_2 = u} (au_1) \otimes (u_2 \star bv) \\
&+ \sum_{u_1 u_2 = u, v_1 v_2 = v} b(au_1 \star v_1) \otimes (u_2 \star v_2) + \sum_{v_1 v_2 = v} (bv_1) \otimes (au \star v_2) \\
&+ [a, b] \otimes 1_{X^*} \sum_{\substack{u_1 u_2 = u \\ v_1 v_2 = v}} (u_1 \otimes u_2) \star (v_1 \otimes v_2) \\
&+ (1_{X^*} \otimes a(u \star bv) + 1_{X^*} \otimes b(au \star v) + 1_{X^*} \otimes [a, b](u \star v)) \\
&= (a \otimes 1_{X^*})(\Delta(u) \star \Delta(w_2)) + 1_{X^*} \otimes a(u \star bv) + (b \otimes 1_{X^*})(\Delta(w_1) \star \Delta(v)) \\
&+ 1_{X^*} \otimes b(au \star v) + ([a, b] \otimes 1_{X^*})(\Delta(u) \star \Delta(v)) + 1_{X^*} \otimes [a, b](u \star v).
\end{aligned}$$

Using the induction hypothesis then the lemma 2.1 (since $[a, b] \in AX$) gives

$$\begin{aligned}
\Delta(w_1) \star \Delta(w_2) &= \Delta(a(u \star w_2)) + \Delta(b(w_1 \star v)) + \Delta([a, b](u \star v)) \\
&= \Delta(a(u \star w_2) + b(w_1 \star v) + [a, b](u \star v)) \\
&= \Delta(w_1 \star w_2).
\end{aligned}$$

\square

Remark 2.2 In particular, Δ is a \sqcup -homomorphism, a \boxplus -homomorphism and a \boxdot -homomorphism.

Let \mathcal{C}_n be the set of positive integer sequences (i_1, \dots, i_k) such that $i_1 + \dots + i_k = n$.

Theorem 2.2 Define a_\star by, for all x_1, \dots, x_n in X ,

$$a_\star(x_1 \dots x_n) = \sum_{(i_1, \dots, i_k) \in \mathcal{C}_n} (-1)^k x_1 \dots x_{i_1} \star x_{i_1+1} \dots x_{i_1+i_2} \star \dots \star x_{i_1+\dots+i_{k-1}+1} \dots x_n$$

then, if $\star \in \mathcal{P}$, $(A\langle X \rangle, \star, \Delta, \epsilon, a_\star)$ is a Hopf algebra.

Proof. With the applications :

$$\begin{array}{ccc} \mu : A & \rightarrow & A\langle X \rangle \\ \lambda & \mapsto & \lambda 1_{X^\star} \end{array} \quad \text{and} \quad \begin{array}{ccc} m : A\langle X \rangle \otimes A\langle X \rangle & \rightarrow & A\langle X \rangle \\ u \otimes v & \mapsto & u \star v \end{array},$$

the antipode must verify $m \circ (a_\star \otimes Id) \circ \Delta = \mu \circ \epsilon$, or, in equivalent terms

$$\sum_{uv=w} a_\star(u) \star v = \langle w | 1_{X^\star} \rangle 1_{X^\star}.$$

$$\text{i.e. } \begin{cases} a_\star(1_{X^\star}) = 1_{X^\star n} \\ \forall x \in X, a_\star(x) = -x \end{cases} \quad \text{and, if } w = x_1 \dots x_n \text{ with } n \geq 2, x_1, \dots, x_n \in X,$$

$$a_\star(w) = - \sum_{k=1}^{n-1} a_\star(x_1 \dots x_k) \star x_{k+1} \dots x_n.$$

An induction over the length n shows that a_\star defined in theorem verifies these equalities, and, in the same way, a_\star verifies $m \circ (Id \otimes a_\star) \circ \Delta = \mu \circ \epsilon$. \square

Corollary 2.1 If \star is \sqcup or \boxplus or \boxdot or \boxtimes , then this construction gives an Hopf algebra. Moreover, for \sqcup or \boxplus , we obtain a graduated Hopf algebra.

2.2 Iterated integral

Let us associate to each letter x_i in X a 1-differential form ω_i , defined in some connected open subset \mathcal{U} of \mathbb{C} . For all paths $z_0 \rightsquigarrow z$ in \mathcal{U} , the *Chen iterated integral* associated to $w = x_{i_1} \dots x_{i_k}$ along $z_0 \rightsquigarrow z$, noted is defined recursively as follows

$$\alpha_{z_0}^z(w) = \int_{z_0 \rightsquigarrow z} \omega_{i_1}(z_1) \alpha_{z_0}^{z_1}(x_{i_2} \dots x_{i_k}) \quad \text{and} \quad \alpha_{z_0}^z(1_{X^\star}) = 1, \quad (2)$$

verifying the *rule of integration by parts* [2] :

$$\alpha_{z_0}^z(u \sqcup v) = \alpha_{z_0}^z(u) \alpha_{z_0}^z(v). \quad (3)$$

We extended this definition over $A\langle X \rangle$ (resp. $A\langle\langle X \rangle\rangle$) by

$$\alpha_{z_0}^z(S) = \sum_{w \in X^\star} \langle S | w \rangle \alpha_{z_0}^z(w). \quad (4)$$

2.3 Shuffle relations

2.3.1 First encoding for colored Hurwitz polyzêtas

Let $\xi = (\xi_n)$ be a sequence of complex numbers and T a family of parameters. Put X' an alphabet indexed over $\mathbb{N}^* \times \mathbb{C}^{\mathbb{N}} \times T$ and $X = \{x_0\} \cup X'$. To each x in X we associate the differential form :

$$\begin{cases} \omega_0(z) = \frac{dz}{z} & \text{si } x = x_0 \\ \omega_{i,\xi,t}(z) = \frac{\prod_{k=1}^i \xi_k}{1 - \prod_{k=1}^i \xi_k z} \times \frac{dz}{z^t} & \text{if } x = x_{i,\xi,t} \text{ with } i \geq 1. \end{cases} \quad (5)$$

For any T -tuple $\mathbf{t} = (t_1, \dots, t_r)$ we associate the T -tuple $\bar{\mathbf{t}} = (\bar{t}_1, \dots, \bar{t}_r)$ given by

$$\begin{cases} \bar{t}_1 = t_1 - t_2, \\ \bar{t}_2 = t_2 - t_3, \\ \vdots \\ \bar{t}_r = t_{r-1} - t_r \end{cases} \quad \text{in this way} \quad \begin{cases} t_1 = \bar{t}_1 + \bar{t}_2 + \dots + \bar{t}_r, \\ t_2 = \bar{t}_2 + \dots + \bar{t}_r, \\ \vdots \\ t_r = \bar{t}_r \end{cases} \quad (6)$$

We choose the sequence ξ and the family \mathbf{t} such that the condition (E) is satisfied.

Proposition 2.4 *For any $\mathbf{s} = (s_1, \dots, s_r)$ with $s_1 > 1$ if $\xi = (\xi_1, \dots, \xi_r) \in \mathcal{E}^r$ and $\mathbf{t} = (t_1, \dots, t_r) \in T^r$, then $\text{Di}(\mathbf{F}_{\xi,\mathbf{t}}; \mathbf{s}) = \alpha_0^1(x_0^{s_1-1} x_{1,\xi,\bar{t}_1} \dots x_0^{s_r-1} x_{r,\xi,\bar{t}_r})$.*

Proof. Since $\omega_{i,\xi,t}(z) = \sum_{n>0} \prod_{k=1}^i \xi_k^n \frac{z^n dz}{z^{1+t}}$ then $\alpha_0^z(x_{r,\xi,\bar{t}_r}) = \sum_{n>0} \prod_{k=1}^r \xi_k^n \frac{z^{n-\bar{t}_r}}{n-\bar{t}_r}$ and $\alpha_0^z(x_0^{s_r-1} x_{r,\xi,\bar{t}_r}) = \sum_{n>0} \prod_{k=1}^r \xi_k^n \frac{z^{n-\bar{t}_r}}{(n-\bar{t}_r)^{s_r}}$. Hence, $\alpha_0^1(x_0^{s_1-1} x_{1,\xi,\bar{t}_1} \dots x_0^{s_r-1} x_{r,\xi,\bar{t}_r})$ gives $\sum_{m_1, \dots, m_r > 0} \prod_{j=1}^r \frac{\prod_{k=1}^j \xi_{k_j}^{m_j}}{(m_j + \dots + m_r - \bar{t}_j - \dots - \bar{t}_r)^{s_j}}$, and then, by change of variables, $\sum_{n_1 > \dots > n_r > 0} \frac{\xi_1^{n_1} \dots \xi_r^{n_r}}{(n_1 - t_1)^{s_1} \dots (n_r - t_r)^{s_r}}$. \square

Theorem 2.3 *Let \mathcal{T} be the group of parameters generated by $\langle T; + \rangle$, \mathcal{C} be a sub-group of (\mathbb{C}^*, \cdot) and A a sub-ring of \mathbb{C} . Put $\mathcal{C}' = \mathbb{C}^{\mathbb{N}} \cap \mathcal{E}$ and \mathcal{T}' the set of finite tuple with elements in \mathcal{T} . Then the A algebra generated by $\{\text{Di}(\mathbf{F}_{\xi,\mathbf{t}}; \mathbf{s})\}_{\xi \in \mathcal{C}', \mathbf{t} \in \mathcal{T}'}$ is the A modulus generated by $\{\text{Di}(\mathbf{F}_{\xi,\mathbf{t}}; \mathbf{s})\}_{\xi \in \mathcal{C}', \mathbf{t} \in \mathcal{T}'}$.*

Proof. We have express the product $\text{Di}(\mathbf{F}_{\xi,\mathbf{t}}; \mathbf{s}) \text{Di}(\mathbf{F}_{\xi',\mathbf{t}'}; \mathbf{s}')$, with $\mathbf{s} = (s_1, \dots, s_r)$, $\mathbf{s}' = (s'_1, \dots, s'_{r'})$, $\xi, \xi' \in \mathcal{C}'$ and $\mathbf{t} = (t_1, \dots, t_r)$, $\mathbf{t}' = (t'_1, \dots, t'_{r'}) \in \mathcal{T}'$, as linear combination of colored Hurwitz polyzêtas. This is an iterated integral associated to $x_0^{s_1-1} x_{1,\xi,\bar{t}_1} \dots x_0^{s_r-1} x_{r,\xi,\bar{t}_r} \sqcup x_0^{s'_1-1} x_{1,\xi',\bar{t}'_1} \dots x_0^{s'_{r'}-1} x_{r',\xi',\bar{t}'_{r'}}$ which is a sum of

terms of the form $x_0^{s''-1} x_{1,\xi^{(1)},\overline{t_{(1)}}} \dots x_0^{s''-1} x_{j_i,\xi^{(i)},\overline{t_{(i)}}} \dots x_0^{s''-1} x_{j_{r''},\xi^{(r'')},\overline{t_{(r'')}}$, with $s''_i \in \mathbb{N}$, $\xi^{(i)}$ is ξ or ξ' and $t_{(i)}$ is t_{j_i} or t'_{j_i} for all i ; and $r'' = r + r''$. Note that

$$\begin{aligned}
& \alpha_0^z(x_{i,\xi,\overline{t_i}} x_0^{s-1} x_{j,\xi',\overline{t_j}}) \\
&= \int_0^z \sum_{m>0} \prod_{k=1}^i \xi_k^m z_1^{m-\overline{t_i}-1} dz_1 \int_0^{z_1} \frac{dz_2}{z_2} \dots \int_0^{z_{s+1}} \sum_{n>0} \prod_{k=1}^i \xi_k'^n z_{s+1}^{n-\overline{t_j}-1} dz_{s+1} \\
&= \sum_{m,n>0} \frac{(\xi_1 \dots \xi_i)^m (\xi'_1 \dots \xi'_j)^n}{(m+n-\overline{t_i}-\overline{t_j})(n-\overline{t_j})^s} z^{n+m}, \\
& \alpha_0^1(x_0^{s''-1} x_{1,\xi^{(1)},\overline{t_{(1)}}} \dots x_0^{s''-1} x_{j_i,\xi^{(i)},\overline{t_{(i)}}} \dots x_0^{s''-1} x_{j_{r''},\xi^{(r'')},\overline{t_{(r'')}}) \\
&= \sum_{m_1, \dots, m_{r''}>0} \prod_{i=1}^{r''} \frac{(\xi_1^{(i)} \dots \xi_{j_i}^{(i)})^{m_i}}{(m_i + \dots + m_{r''} - \overline{t_{(i)}} - \dots - \overline{t_{(r'')}})^{s''_i}} \\
&= \sum_{n_1 > \dots > n_{r''}>0} \prod_{i=1}^{r''} \frac{\xi_i^{n_i}}{(n_i - t''_i)^{s''_i}}
\end{aligned}$$

with $n_i = m_i + \dots + m_{r''}$, $t''_i = \overline{t_{(i)}} + \dots + \overline{t_{(r'')}}$ for all i , so $\mathbf{t}'' \in \mathcal{T}$; $\xi''_1 = \xi_1^{(1)}$ and $\xi''_i = \frac{\xi_1^{(i)} \dots \xi_{j_i}^{(i)}}{\xi_1^{(i-1)} \dots \xi_{j_{i-1}}^{(i-1)}}$ for $i > 1$ so $\xi'' \in \mathcal{C}$: we can express each term of the shuffle product as $\text{Di}(\mathbf{F}_{\xi'', \mathbf{t}''}; \mathbf{s}'')$. \square

Note that the shuffle product over two words of X^*X' acts separately over (\mathcal{C}', \cdot) , $(\mathcal{T}', +)$ and the convergent compositions. We can describe the situation with the shuffle algebra⁵:

Theorem 2.4 *Let \mathcal{H} be the \mathbb{Q} -algebra generated by the colored Hurwitz polyzêtas. The map $\zeta : (\mathbb{Q}\langle (x_0^* x_{i,\xi,t})^* \rangle, \sqcup) \rightarrow (\mathcal{H}, \cdot)$, $x_0^{s_1} x_{1,\xi,\overline{t_1}} \dots x_0^{s_r} x_{r,\xi,\overline{t_r}} \mapsto \text{Di}(\mathbf{F}_{\xi,t}; \mathbf{s} + 1)$ is a surjective algebra morphism.*

Example 7 *Since $\text{Di}(\mathbf{F}_{\xi,t}; 3) = \alpha_0^1(x_0^2 x_{1,\xi,t})$ and $\text{Di}(\mathbf{F}_{\xi',t'}; 2) = \alpha_0^1(x_0 x_{1,\xi',t'})$ then $\text{Di}(\mathbf{F}_{x_i,t}; 3) \text{Di}(\mathbf{F}_{x_{i'},t'}; 2) = \alpha_0^1(x_0 x_{1,\xi',t'} \sqcup x_0^2 x_{1,\xi,t})$. Example 1 with $x = x_{1,\xi,t}$ and $x' = x_{1,\xi',t'}$ gives the expression of $x_0 x_{1,\xi',t'} \sqcup x_0^2 x_{1,\xi,t}$. But the first term obtained is*

$$\begin{aligned}
& \alpha_0^1(x_0 x_{1,\xi',t'} x_0^2 x_{1,\xi,t}) \\
&= \int_0^1 \frac{dz_1}{z_1} \int_0^{z_1} \sum_{m>0} \xi'^m z_2^{m-t'-1} dz_2 \int_0^{z_2} \frac{dz_3}{z_3} \int_0^{z_3} \frac{dz_4}{z_4} \int_0^{z_4} \sum_{n>0} \xi^n z_5^{n-t-1} dz_5 \\
&= \sum_{n,m>0} \frac{\xi'^m \xi^n}{(m+n-t'-t)^2 (n-t)^3} \\
&= \sum_{n_1 > n_2 > 0} \frac{(\xi')^{n_1} (\xi/\xi')^{n_2}}{(n_1 - t' - t)^2 (n_2 - t)^3} \\
&= \text{Di}(\mathbf{F}_{(\xi,\xi/\xi'); (t+t',t)}; (2,3)).
\end{aligned}$$

⁵Working in $\mathbb{Q}\langle (x_0^* x_{i,\xi,t})^* \rangle$ implies working in the graduated Hopf algebra $(\mathbb{Q}\langle X^* \rangle, \sqcup, \Delta, \epsilon, a_{\sqcup})$.

We can make similar calculus for the other terms and find :

$$\begin{aligned}
& \text{Di}(\mathbf{F}_{\xi,t}; 3) \text{Di}(\mathbf{F}_{\xi',t'}; 2) \\
&= \text{Di}(\mathbf{F}_{(\xi', \xi/\xi'); (t+t', t)}; (2, 3)) + 2 \text{Di}(\mathbf{F}_{(\xi', \xi/\xi'); (t+t', t)}; (3, 2)) \\
&\quad + 3 \text{Di}(\mathbf{F}_{(\xi', \xi/\xi'); (t+t', t)}; (4, 1)) + 3 \text{Di}(\mathbf{F}_{(\xi, \xi'/\xi); (t+t', t')}; (4, 1)) \\
&\quad + \text{Di}(\mathbf{F}_{(\xi, \xi'/\xi); (t+t', t')}; (3, 2)).
\end{aligned}$$

2.3.2 Second encoding for colored Hurwitz polyzêtas

For the Hurwitz polyzêtas, we can obtain an encoding indexed by a finite alphabet. Let the alphabet $X = \{x_0; x_1\}$ and associate to x_0 the form $\omega_0(z) = z^{-1}dz$ and at x_1 the form $\omega_1(z) = (1-z)^{-1}dz$.

For each $x \in X$ and $\lambda \in \mathbb{C}$, we note $(\lambda x)^* = \sum_{k \geq 0} (\lambda x)^k$. Then, (see [10], [11]), $\alpha_0^1(x_0^{s_1-1}(t_1 x_0)^{s_1} x_1 \dots x_0^{s_r-1}(t_r x_0)^{s_r} x_1) = \zeta(\mathbf{s}; \mathbf{t})$.

Theorem 2.5 *Let \mathcal{H}' be the \mathbb{Q} -algebra generated by the Hurwitz polyzêtas and \mathcal{X} the \mathbb{Q} -algebra generated by $(t_1 x_0)^{s_1} x_1 \dots (t_r x_0)^{s_r} x_r$. Then, $\zeta : (\mathcal{X}, \sqcup) \rightarrow (\mathcal{H}', \cdot)$ is a surjective morphism of algebras.*

Note that we can apply the idea of encoding of “simple” colored Hurwitz zetas functions (with depth one : $r = 1$). Let $\xi = (\xi_n)$ be a sequence of complex numbers in the unit ball $\mathcal{B}(0; 1)$ and T a family of parameters. Let $X = \{x_0, x_1, \dots\}$ be a alphabet indexed by \mathbb{N} . Associate to x_0 the differential form $\omega_0(z) = z^{-1}dz$ and to $x_i, i \geq 1$, the differential form $\omega_i(z) = \xi_i(1 - \xi_i z)^{-1}dz$.

Proposition 2.5 *With this notation, $\alpha_0^1(((tx_0)^* x_0)^{s-1} (tx_0)^* x_i) = \sum_{n > 0} \frac{\xi_i^n}{(n-t)^s}$.*

Proof. Since $\frac{\xi_i dz_0}{1 - \xi_i z_0} = \xi_i \sum_{n \geq 0} (\xi_i z_0)^n dz_0$, we can write

$$\alpha_0^z((tx_0)^k x_i) = t^k \int_0^z \frac{dz_k}{z_k} \int_0^{z_k} \dots \int_0^{z_1} \xi_i \sum_{n \geq 0} (\xi_i z_0)^n dz_0 = \sum_{n > 0} t^k \frac{\xi_i^n z^n}{n^{k+1}},$$

for $z \in \mathcal{B}(0; 1)$ and for $k \in \mathbb{N}$. Thanks to the absolute convergence,

$$\alpha_0^z((tx_0)^* x_i) = \sum_{n > 0} \frac{\xi_i^n z^n}{n} \sum_{k \geq 0} \left(\frac{t}{n}\right)^k = \sum_{n > 0} \frac{\xi_i^n z^n}{n-t}.$$

In the same way, if $z \in \mathcal{B}(0; 1)$:

$$\begin{aligned}
\forall k \in \mathbb{N}, \quad \alpha_0^z((tx_0)^k x_0 (tx_0)^* x_i) &= \sum_{n > 0} t^k \frac{\xi_i^n}{n-t} \frac{z^n}{n^{k+1}}, \\
\text{so} \quad \alpha_0^z((tx_0)^* x_0 (tx_0)^* x_i) &= \sum_{n > 0} \frac{\xi_i^n z^n}{(n-t)^2}
\end{aligned}$$

$$\text{and} \quad \alpha_0^z \left(((tx_0)^* x_0)^{s-1} (tx_0)^* x_i \right) = \sum_{n>0} \frac{\xi_i^n z^n}{(n-t)^s}.$$

□

Remark 2.3 Note that, with the same notation,

$$\begin{aligned} \alpha_0^z \left(x_1 ((t_2 x_0)^* x_0)^{s-1} (t_2 x_0)^* x_2 \right) &= \sum_{n,m>0} \frac{\xi_2^n \xi_1^m z^{n+m}}{(n-t_2)^s (m+n)} \\ &= \sum_{n_1>n_2>0} \frac{\xi_2^{n_2} \xi_1^{n_1-n_2} z^{n_1}}{n_1 (n_2-t_2)^s}. \end{aligned}$$

In other words, this encoding appears to be widespread only as couples of the type

$$\xi = (1, 1, \dots, 1, \xi_r) : \text{with } \xi_1 = 1 \text{ and } \omega_1 = (1-z)^{-1} dz,$$

$$\begin{aligned} &\alpha_0^1 \left(x_0^{s_1-1} (t_1 x_0)^{*s_1} x_1 \dots x_0^{s_{r-1}-1} (t_{r-1} x_0)^{*s_{r-1}} x_{r-1} x_0^{s_r-1} (t_r x_0)^{*s_r} x_r \right) \\ &= \sum_{n_1>\dots>n_r} \frac{\xi_r^{n_r}}{(n_1-t_1)^{s_1} \dots (n_r-t_r)^{s_r}}. \end{aligned}$$

2.4 Duffle relations

Let $\lambda = (\lambda_n)$ be a set of parameters, $\mathbf{s} = (s_1, \dots, s_r)$ a composition, $\xi \in \mathbb{C}^r$. Then

$$\forall n \in \mathbb{Z}_{>0}, \quad M_{\mathbf{s}, \xi}^n(\lambda) = \sum_{n>n_1>\dots>n_r>0} \prod_{i=1}^r \xi_i^{n_i} \lambda_{n_i}^{s_i} \quad \text{and} \quad M_{(), ()}^n(\lambda) = 1. \quad (7)$$

We can export the duffle over the tuples $\mathbf{s} = (s_1, \dots, s_r) \in \mathbb{Z}_{>0}^r$ and $\xi \in \mathbb{C}^r$ with :

$$\begin{aligned} &(\mathbf{s}, \xi) \boxtimes ((), 1) = ((), 1) \boxtimes (\mathbf{s}, \xi) = (\mathbf{s}, \xi) \quad \text{and} \\ &(s_1, \mathbf{s}; \xi_1, \xi) \boxtimes (r_1, \mathbf{r}; \rho_1, \rho) \\ &= (s_1; \xi_1). ((\mathbf{s}; \xi) \boxtimes (r_1, \mathbf{r}; \rho_1, \rho)) + (r_1; \rho_1). ((s_1, \mathbf{s}; \rho_1, \xi) \boxtimes (\mathbf{r}; \rho)) \\ &\quad + (s_1 + r_1; \xi_1 \rho_1). ((\mathbf{s}; \xi) \boxtimes (\mathbf{r}; \rho)) \end{aligned} \quad (8)$$

Proposition 2.6 Let $\mathbf{s} = (s_1, \dots, s_l)$ and $\mathbf{r} = (r_1, \dots, r_k)$ be two compositions, $\xi \in \mathbb{C}^l$, $\rho \in \mathbb{C}^k$. Then

$$\forall n \in \mathbb{N}, \quad M_{\mathbf{s}, \xi}^n(\lambda) M_{\mathbf{r}, \rho}^n(\lambda) = M_{(\mathbf{s}, \xi) \boxtimes (\mathbf{r}, \rho)}^n(\lambda).$$

Proof. Put the compositions $\mathbf{s}' = (s_2, \dots, s_l)$, $\mathbf{r}' = (r_2, \dots, r_k)$, the tuples of complex numbers $\xi' = (\xi_2, \dots, \xi_l)$ and $\rho' = (\rho_2, \dots, \rho_k)$, then

$$\begin{aligned} &M_{\mathbf{s}, \xi}^n(\lambda) M_{\mathbf{r}, \rho}^n(\lambda) \\ &= \sum_{n>n_1, n>n'_1} \xi_1^{n_1} \lambda_{n_1}^{s_1} M_{\mathbf{s}', \xi'}^{n_1}(\lambda) \rho_1^{n'_1} \lambda_{n'_1}^{r_1} M_{\mathbf{r}', \rho'}^{n'_1}(\lambda) \\ &= \sum_{n>n_1} \xi_1^{n_1} \lambda_{n_1}^{s_1} M_{\mathbf{s}', \xi'}^{n_1}(\lambda) M_{\mathbf{r}, \rho}^{n_1}(\lambda) + \sum_{n>n'_1} \rho_1^{n'_1} \lambda_{n'_1}^{r_1} M_{\mathbf{s}, \xi}^{n'_1}(\lambda) M_{\mathbf{r}', \rho'}^{n'_1}(\lambda) \end{aligned}$$

$$+ \sum_{n>m} (\xi_1 \rho_1)^m \lambda_m^{s_1+r_1} M_{\mathbf{s}', \xi'}^m(\lambda) M_{\mathbf{r}', \rho'}^m(\lambda).$$

A recurrence ended the demonstration. \square

Theorem 2.6 Let $\mathbf{s} = (s_1, \dots, s_l)$ and $\mathbf{r} = (r_1, \dots, r_k)$ be two compositions, ξ a l -tuple and ρ a k -tuple of \mathcal{E} , $\mathbf{t} = (t, \dots, t)$ a l -tuple and $\mathbf{t}' = (t, \dots, t)$ a k -tuple, both formed by the same parameter t diagonally. Then

$$\text{Di}(\mathbf{F}_{\xi, \mathbf{t}; \mathbf{s}}) \text{Di}(\mathbf{F}_{\xi', \mathbf{t}'; \mathbf{s}'} = \text{Di}(\mathbf{F}_{\xi'', (t, \dots, t); \mathbf{s}''}),$$

with $(\mathbf{s}''; \xi'') = (\mathbf{s}; \xi) \boxtimes (\mathbf{s}'; \xi')$.

Proof. With $\lambda_n = \frac{1}{n-t}$ for all $n \in \mathbb{N}$, $M_{\mathbf{s}, \xi}^n(\lambda) = \sum_{n>n_1>\dots>n_r} \prod_{i=1}^r \frac{\xi_i^{n_i}}{(n_i-t)^{s_i}}$. So $\lim_{n \rightarrow \infty} M_{\mathbf{s}, \xi}^n(\lambda) = \text{Di}(\mathbf{F}_{\xi, \mathbf{t}; \mathbf{s}})$ and taking the limit of Proposition 2.6 gives the result. \square

Example 8 The use of examples 2 and 4 gives

$$\begin{aligned} & \text{Di}(\mathbf{F}_{(\frac{2}{3}, -1), \mathbf{t}; (3, 1)}) \text{Di}(\mathbf{F}_{(\frac{1}{2}, t); (2)}) \\ &= \text{Di}(\mathbf{F}_{(\frac{2}{3}, -1, \frac{1}{2}), (t, t, t); (3, 1, 2)}) + \text{Di}(\mathbf{F}_{(\frac{2}{3}, \frac{1}{2}, -1), (t, t, t); (3, 2, 1)}) \\ &+ \text{Di}(\mathbf{F}_{(\frac{2}{3}, -\frac{1}{2}), \mathbf{t}; (3, 3)}) + \text{Di}(\mathbf{F}_{(\frac{1}{2}, \frac{2}{3}, -1), (t, t, t); (2, 3, 1)}) + \text{Di}(\mathbf{F}_{(\frac{1}{3}, -1), \mathbf{t}; (5, 1)}) \end{aligned}$$

Remark 2.4 Extend the duffle product to triplets $(\mathbf{s}, \mathbf{t}, \xi) \in \cup_{r \in \mathbb{N}^*} \mathbb{N}^r \times \{t\}^r \times \mathbb{C}^r$ by

$$\begin{aligned} (s_1, \mathbf{s}; t, \mathbf{t}; \xi_1, \xi) \boxtimes (r_1, \mathbf{r}; t, \mathbf{t}'; \rho_1, \rho) &= (s_1; t; \xi_1). ((\mathbf{s}; \mathbf{t}; \xi) \boxtimes (r_1, \mathbf{r}; t, \mathbf{t}'; \rho_1, \rho)) \\ &+ (r_1; t; \rho_1). ((s_1, \mathbf{s}; t, \mathbf{t}; \rho_1, \xi) \boxtimes (\mathbf{r}; \mathbf{t}'; \rho)) \\ &+ (s_1 + r_1; t; \xi_1 \rho_1). ((\mathbf{s}; \mathbf{t}; \xi) \boxtimes (\mathbf{r}; \mathbf{t}'; \rho)), \end{aligned}$$

and define the function F over $\mathcal{I} = \cup_{r \in \mathbb{N}^*} \mathbb{N}^r \times \{t\}^r \times \mathbb{C}^r$ by $F(\mathbf{s}, \mathbf{t}, \xi) = \text{Di}(\mathbf{F}_{\xi, \mathbf{t}; \mathbf{s}})$. Then, by Theorem 2.6, the function $F : (\mathcal{I}, \boxtimes) \rightarrow (\mathbb{C}, \cdot)$ is morphism of algebras.

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